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CLASSROOM NOTES

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This section contains brief notes of up to five typed pages. Articles illustrate novel ideas and insights related to either the teaching of mathematics for applications or the teaching of the applications.

A MODEL OF DIETING*

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Abstract. We construct a mathematical model of the dieting process and investigate the properties of its solutions. While the general solution cannot be explicitly obtained, we show that all the important features can be determined by use of geometrical techniques and linear stability analysis. The results of this study are consistent with all accepted facts relating to the dieting process.

Key words. dieting, mathematical modeling, differential equations

AMS subject classifications. 34A26, 34D99, 92B05

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1. Introduction.

Dieting as a process and social phenomenon is a multibillion dollar industry in the United States. This industry deals with the following products and services: (i) diet foods and nutritional supplements; (ii) sports and dieting clothing and equipment; (iii) exercise tapes, videos, and television programs; and (iv) articles, books, and other publications on how to achieve weight reduction. However long-term weight loss and its maintenance is very difficult for most individuals [1, 4]. A variety of complex personal and physiological factors are the basis for this situation [1, 4, 5].

The time line for most individuals who lose weight, i.e., who through dieting attempt to achieve their desired weight level, goes as follows:

(a) A decision is made to reduce body weight by a certain amount.
(b) A diet and physical exercise plan is selected.
(c) Initially, there is rapid weight loss.
(d) Eventually, the desired weight is attained.
(e) The weight loss is maintained for a (generally) short period of time.
(f) The individual returns to essentially the same "eating pattern" as before the weight was lost.
(g) Rather quickly, the weight increases with often a final body weight that is larger than the initial (prediet) weight.

Our main purpose is to construct a rather straightforward mathematical model of the dieting process and investigate the properties of its solutions. The model

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is given in terms of a nonlinear, first-order, ordinary differential equation. While this equation cannot be solved exactly, in terms of elementary functions, we show that all the important features of its solutions can be determined by use of phase-space (geometrical) techniques and linear stability analysis [3]. A second and equally important goal is to show that mathematical modeling techniques can be used to provide important insight and guidance for a biomedical problem of great interest to the general public. A third feature of this paper is to show that the value of a mathematical model may not come from giving exact numerical and/or analytical results, but by providing qualitative knowledge of the possible solutions of relevance to the system.

2. The model. Let $W(t)$ denote the body weight of an individual at time t. Our main assumption is that the rate of change of $W(t)$, with respect to time, is given by the relation

$$\frac{dW}{dt} = F(t) - E(t) - M(W), \tag{1}$$

where $F(t)$ is a food intake function, $E(t)$ is a function that provides a measure of the physical activity done by the individual, and $M(W)$ is a body metabolism function which depends on W. Note that $F(t)$ is a nonnegative function which measures the caloric intake of the dieter. $E(t)$ is also a nonnegative function and is a measure of the nonmetabolic energy needs of the body. The dominant component to this function is physical exercise.

The function $M(W)$ is nonnegative. All available data are consistent with it having the functional form [2]

$$M(W) \sim W^{\alpha}, \qquad \alpha \approx 0.7. \tag{2}$$

For our purposes, the following function is selected:

$$M(W) = \beta W^{3/4}, \tag{3}$$

where β is a positive parameter. A more complex expression for $M(W)$ that is consistent with (2) is

$$M(W) = \frac{AW}{1 + BW^{1-\alpha}}, \qquad 0 < \alpha < 1, \tag{4}$$

where A and B are positive parameters. Use of (4) does not change any of the critical conclusions of the paper; however, the details of the analysis become complex.

The substitution of (3) into (1) gives the differential equation

$$\frac{dW}{dt} = F(t) - E(t) - \beta W^{3/4}. \tag{5}$$

This equation provides a mathematical model of the dieting process. The central question is what are the functions $F(t)$ and $E(t)$?

3. Dieting strategy. In general, a diet consists of selecting a strategy or plan for the determination of the amount of food (calories) to be eaten during a given interval of time and the determination of the number of calories to be expended in directed physical activity during this same interval of time. The *simplest* strategy is to let both the food intake and exercise regiment be constant over a convenient and meaningful time interval. For example, the caloric intake can be selected to be $\bar{F}$ calories/day from food with $\bar{E}$ calories/day "burned" in exercise. Both $\bar{F}$ and $\bar{E}$ are assumed to be constant. Note that the reference time interval is one day. This particular diet strategy leads to the result

$$[F(t) - E(t)]_{av} = \bar{F} - \bar{E} = \lambda, \tag{6}$$

where the notation means that the left side of (6) is "averaged" over one day. In general, the parameter λ is nonnegative, i.e., it is difficult to imagine any physiological state of a normal human for which λ would be negative.

Substituting the diet strategy given by (6) into (5) leads to our model equation

$$\frac{dW}{dt} = \lambda - \beta W^{3/4}. \tag{7}$$

This is a nonlinear, first-order differential equation that depends on two parameters, λ and β.

4. Analysis of the model. The long-time behavior of the solutions to (7) is determined by the fixed-points or constant solutions. This equation has a single fixed-point at the value

$$\bar{W} = \left(\frac{\lambda}{\beta}\right)^{4/3}. \tag{8}$$

An application of linear stability analysis [3] will allow the determination of the behavior of the solutions in a small neighborhood of the fixed-point. To proceed, let

$$W(t) = \bar{W} + \epsilon(t), \qquad |\epsilon(0)| \ll \bar{W}, \tag{9}$$

where $\epsilon(0)$ is a small perturbation from the fixed-point. Then substituting (9) into (7) and retaining only linear terms in $\epsilon(t)$ gives

$$\frac{d\epsilon}{dt} = -R\epsilon, \qquad R = \left(\frac{3\beta}{4}\right)\left(\frac{\beta}{\lambda}\right)^{1/3} > 0. \tag{10}$$

Since $R > 0$, it follows that

$$\operatorname*{Lim}_{t\to\infty} \epsilon(t) = 0 \tag{11}$$

and the fixed-point $W(t) = \bar{W}$ is linearly stable. Consequently, for fixed λ and β, small perturbations in the body weight will die out.

The global behavior of the solutions for positive initial data, i.e., $W(0) > 0$, can be found by examining (7). Now,

$$\frac{dW}{dt} = \begin{cases} < 0, & W > \bar{W}, \\ > 0, & 0 < W < \bar{W}. \end{cases} \tag{12}$$

Hence, it can be concluded that all solutions monotonically approach the fixed-point with increase in time [3], i.e.,

$$\operatorname*{Lim}_{t\to\infty} W(t) = \bar{W}, \qquad W(0) > 0. \tag{13}$$

This means that the fixed-point is globally stable; see Figure 1.

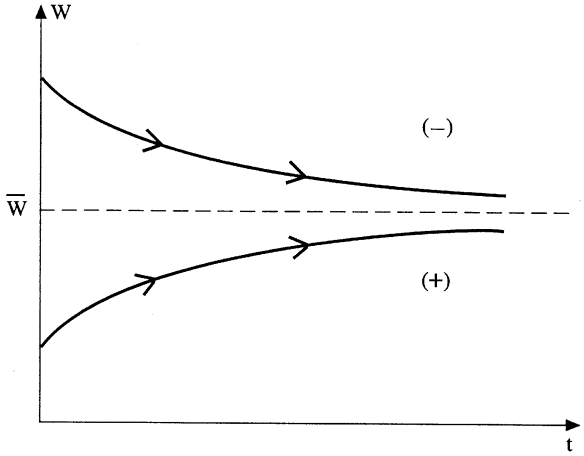

FIG. 1. *Typical solutions of* (7) *for positive initial conditions.*

5. Discussion. Under normal conditions, as stated in section 3, the parameter $\lambda > 0$. The situation where $\lambda \leq 0$ is clearly pathological in the sense that all solutions to (7) tend to zero, i.e.,

$$\lim_{t\to\infty} W(t) = 0, \qquad \lambda \leq 0. \tag{14}$$

This case corresponds to starvation: $\bar{F} = 0$ and $\bar{E} \geq 0$. (For a real person $W(t)$ cannot be negative. Thus, for $\lambda \leq 0$, if for some $t = t_1$, $W(t_1) = 0$, then $W(t) = 0$ for $t > t_1$.) For a given individual β is constant and once a particular diet strategy is selected (by specifying the value of λ) the equilibrium weight $\bar{W}$ is determined. Note, from (8), that $\bar{W}$ is proportional to $\lambda^{4/3}$ and consequently increases faster than a linear behavior on λ.

Consider the situation where weight loss is desired. Let the initial weight be $\bar{W}_i$ and assume that a smaller weight $\bar{W}_f$ is the goal of the dieter. To achieve this reduction the parameter λ must change from λ_i to λ_f, where

$$\lambda_i = \beta \bar{W}_i^{3/4} > \lambda_f = \beta \bar{W}_f^{3/4}. \tag{15}$$

Figure 2 presents a graphic depiction of this case. The individual maintains weight $\bar{W}_i$ until the time $t = t_0$. At that time the diet begins and the individual's weight decreases monotonically to the new, smaller value $\bar{W}_f$.

If the individual wishes to gain weight, then the reverse of the above procedure is done, i.e., λ_i must increase to λ_f; see Figure 3.

In summary, we have constructed a mathematical model of the dieting process. The model is formulated in terms of a nonlinear, first-order ordinary differential equation. The model depends on two parameters, β and λ. For a given individual β is constant. The dieting strategy is determined by selecting a (positive) value for λ. Each value of λ corresponds to a unique final body weight as given in (8). Since $\bar{W}$ increases faster with λ than a linear function, significant changes in the dieter's "life-style" can lead to even larger changes in the value of the final body weight. Note that for a given value of λ there are many combinations of $\bar{F}$ and $\bar{E}$ which give $\lambda = \bar{F} - \bar{E} =$ constant. Thus, a given diet strategy can be achieved by various combinations of caloric (food) restriction and physical exercise. A detailed examination

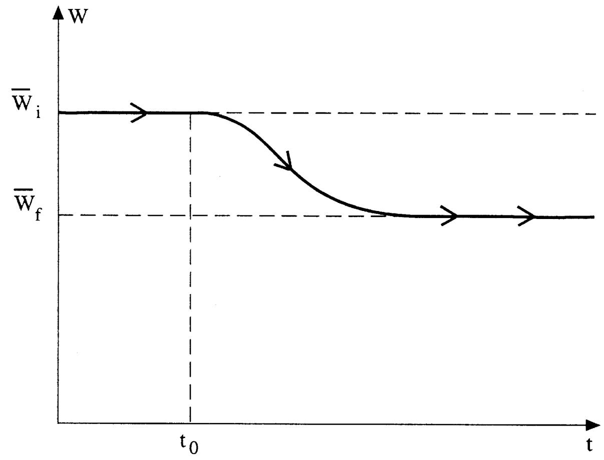

FIG. 2. *Loss of weight diagram. The initial weight is $\bar{W}_i$ and $\bar{W}_f$ is the desired weight, $\bar{W}_i > \bar{W}_f$.*

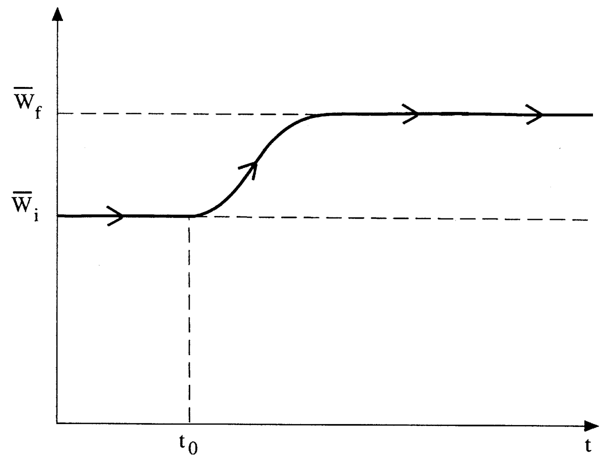

FIG. 3. *Gain of weight diagram. The initial weight is $\bar{W}_i$ and $\bar{W}_f$ is the desired weight, $\bar{W}_i < \bar{W}_f$.*

of the linear stability analysis results indicates that the greatest decrease of weight per unit time occurs at the start of the diet. Likewise, the weight loss per unit time is smallest when the weight goal is nearly accomplished. An important conclusion is that to stay at the desired weight $\bar{W}_f$, the original diet strategy that brought about the weight loss must be maintained "forever"! Finally, it should be indicated that the results of this model are all consistent with current data on dieting and related issues [1, 4].

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